IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

Integral Points On The Homogeneous Cone $Z^2 = 2X^2 + 8Y^2$

M.A. Gopalan[*1], A. Vijayasankar[2]

[*1]Department of Mathematics Shrimati Indira Gandhi College Tiruchirappalli, India

[2]Department of Mathematics National College Tiruchirappalli, India

mayilgopalan@gmail.com

Abstract

The homogeneous cone represented by the ternary quadratic Diophantine equation $z^2 = 2x^2 + 8y^2$ is analyzed for its patterns of non zero distinct integral solutions. A few interesting relations between the solutions, special polygonal numbers, pyramidal numbers and other special numbers are exhibited.

Keywords: Homogeneous Cone, Ternary Quadratic, Integral Solutions. Mathematics subject classification No:11D09..

Introduction

The ternary quadratic diophantine equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-22]. This communication concerns with another interesting ternary quadratic equation $z^2 = 2x^2 + 8y^2$ representing a homogeneous cone for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

Notation Used

- $T_{m,n}$ = Polygonal Number of rank n with sides m
- p_n^m = Pyramidal number of rank n with sides m
- ST_n = Star number of rank n
- j_n = Jacobsthal-Lucas number of rank n
- Ky_n = Kynea number of rank n

Method of Analysis

The homogeneous cone represented by the ternary quadratic equation is

$$z^2 - 8y^2 = 2x^2$$

(1)

Assume $x = x(a,b) = a^2 - 8b^2, a, b \neq 0$

(2)

$$= (a + \sqrt{8}b)(a - \sqrt{8}b)$$

We present below different patterns of integral solutions of (1)

Pattern I

Write 2 as

$$2 = \frac{(4+\sqrt{8})(4-\sqrt{8})}{4}$$

(3)

Substituting (2) and (3) in (1) and employing the method of factorization, define

$$z + \sqrt{8}y = \frac{1}{2}(4+\sqrt{8})(a+\sqrt{8}b)$$

(4)

Equating rational and irrational parts of (4),we have

$$z = z(a,b) = 2(a^2 + 8b^2 + ab)$$

(5)

and

$$y = y(a,b) = \frac{1}{2}(a^2 + 8b^2 + 8ab)$$

(6)

As our interest centers on finding integer solutions, it is possible to choose a and b suitably so that y is an integer. Thus we have the following three sets of solutions to(1).

Case (i)

Let $a = 2A, b = 2B$

The corresponding solutions of (1) are

$$x = x(A,B) = 4(A^2 - 8B^2)$$

$$y = y(A,B) = 2(A^2 + 8B^2 + 8AB)$$
$$z = z(A,B) = 8(A^2 + 8B^2 + AB)$$

Properties

1. $y(2^n,1) + 16(-1)^n - 14 = 2(j_{2n} + 8j_n)$
2. $y(2^{2n},1) - 8 = 2(ky_{2n} + 6j_{2n})$
3. $x(-1,2^{2n}) + y(-1,2^{2n}) = 39 - (ST_{2n} + 10j_{2n} + 22j_{2n})$
4. $6(z(1,B) - 4)$ is a Nasty Number

Case(ii)

Let $a = 2A, b = 2B + 1$

The corresponding solutions of (1) are

$$x = x(A,B) = 4[A^2 - 8B(B+1) - 2]$$
$$y = y(A,B) = 2[A^2 + 8B^2 + 4A + 8B + 8AB + 2]$$
$$z = z(A,B) = 8[A^2 + 8B^2 + 2A + 8B + 4AB + 2]$$

Properties

Each of the following expressions is a Nasty Number

1. $6[z(2A,1) + 4A^2 - 80]$
2. $x(A,1) + y(A,1) + 60$
3. $6[4B^2 - x(1,B) - 8B]$
4. $6[y(-1,B) + 26]$

Case(iii)

Let $a = 2nb$

The corresponding solutions of (1) are

$$x = x(n,b) = 4b^2(n^2 - 2)$$
$$y = y(n,b) = 2b^2(n^2 + 4n + 2)$$
$$z = z(n,b) = 8b^2(n^2 + 2n + 2)$$

Properties

1. $x(n,1) - y(n,1) - T_{6,n} \equiv -5 \pmod 7$
2. $y(n,-1) + z(n,1) - T_{14,n} \equiv -1 \pmod{13}$
3. $y(n,1) + z(n,1) - T_{22,n \equiv -} 13 \pmod{33}$
4. $z(n,1) - x(n,1) - 12T_{3,n} \equiv 4 \pmod{10}$

Pattern II

Instead of (3), write 2 as

$$2 = \frac{(20+\sqrt{8})(20-\sqrt{8})}{196}$$

(7)

Following the similar procedure as in Pattern I, the integral solutions for (1) are as follows:

$$x = x(A,B) = 196[A^2 - 8B^2]$$
$$y = y(A,B) = 14[A^2 + 8B^2 + 40AB]$$
$$z = z(A,B) = 14[20A^2 + 160B^2 + 16AB]$$

Properties

1. $y(A,1) - T_{30,A} \equiv -461 \pmod{573}$

Each of the following expressions is a Nasty Number

2. $6[x(A,1) + 1568]$
3. $6[z(A,1) - y(A,1) - T_{22,A} - 185A - 1872]$

Pattern III

Rewrite (1) as $z^2 - 2x^2 = 8y^2$

(8)

Assume

$$y = y(a,b) = a^2 - 2b^2, a,b \neq 0$$

(9)

$$= (a + \sqrt{2}b)(a - \sqrt{2}b)$$

and write 8 as

$$8 = (4 + 2\sqrt{2})(4 - 2\sqrt{2})$$

(10)

For this case, the values of x and z are given by

$$x = x(a,b) = 2a^2 + 4b^2 + 8ab$$
$$z = z(a,b) = 4a^2 + 8b^2 + 8ab$$

(11)

Thus (9)and(11) represent the non-zero distinct integral solutions of (1).

Properties

1 $6[x(A,1) + 2A^2]$ is a Nasty Number.

2. $y(A,1) + z(A,1) - 10T_{3,A} \equiv 0 \pmod 3$

Pattern IV

Instead of (10), write 8 as

$$8 = \frac{(20+2\sqrt{2})(20-2\sqrt{2})}{49}$$

For this choice, the corresponding non-zero distinct integral solutions of (1) are obtained as

$$x = x(A,B) = 14[A^2 + 2B^2 + 20AB]$$

$$y = y(A,B) = 49[A^2 - 2B^2]$$

$$z = z(A,B) = 4[35A^2 + 70B^2 + 14AB]$$

Properties

1. $x(2A,1) - 112T_{3,A} = 504A + 28$
2. $z(A,1) - y(A,1) - T_{50,A} - T_{60,A} - T_{70,A} - 8T_{3,A} \equiv -106 \pmod{136}$
3. $x(1,B) + y(1,B) + 140T_{3,B} \equiv -287 \pmod{350}$

Remarkable Observation

I. If the non zero integer triple (x_0, y_0, z_0) is any solution of (1), then each of the following four triples also satisfy (1)

Triple 1:

$$(x_{2n}, y_{2n}, z_{2n}) = 5^{2n}(x_0, y_0, z_0)$$

Triple 2:

$$(x_{2n-1}, y_{2n-1}, z_{2n-1}) = 5^{2n-2}(-3x_0 + 8y_0, 2x_0 + y_0, 5z_0)$$

Triple 3:

(x_n, y_0, z_n) where

$$x_n = \beta_n x_0 + \alpha_n z_0$$

$$z_n = 2\alpha_n x_0 + \beta_n z_0$$

Here

$$\alpha_n = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}]$$

$$\beta_n = \frac{1}{2}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}]$$

Triple 4: (x_0, y_n, z_n) where

$$y_n = \beta_{2n-1} y_0 + \alpha_{2n-1} z_0$$

$$z_n = 8\alpha_{2n-1} y_0 + \beta_{2n-1} z_0$$

Here

$$\alpha_{2n-1} = \frac{1}{2\sqrt{8}}[(3+2\sqrt{2})^{2n} - (3-2\sqrt{2})^{2n}]$$

$$\beta_{2n-1} = \frac{1}{2}[(3+2\sqrt{2})^{2n} + (3-2\sqrt{2})^{2n}]$$

II. Employing the solutions (x, y, z) of (1), each of the following expressions among the special polygonal and pyramidal numbers is a perfect square.

1. $2\left(\frac{p_x^5}{T_{3,x}}\right)^2 + 8\left(\frac{p_{2y}^5}{T_{3,2y}}\right)^2$
2. $2\left(\frac{3p_{2x}^3}{T_{3,2x+1}}\right)^2 + 8\left(\frac{p_y^3}{T_{3,y+1}}\right)^2$
3. $2\left(\frac{p_y^5}{T_{3,y}}\right)^2 + 72\left(\frac{p_x^3}{T_{3,x+1}}\right)^2$

Conclusions

To conclude, one may search for other patterns of solutions and their corresponding properties.

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